Name: Solutions Section: ______________

Functions

1. Give the definition of the statement "$f(x)$ is a function".

each input has a single output

2. Let $f(x) = x^2 + x + 1$. Find $f(0)$, $f(1)$, and $f(2)$.

$f(0) = 0^2 + 0 + 1 = 1$

$f(1) = 1^2 + 1 + 1 = 3$

$f(2) = 2^2 + 2 + 1 = 4 + 2 + 1 = 7$

3. Let $f(x) = x^2 + x + 1$. Write down and simplify $f(x+1)$ and $\dfrac{f(1+h) - f(1)}{h}$.

$f(x+1) = (x+1)^2 + (x+1) + 1$

$= x^2 + 2x + 1 + x + 1 + 1$

$= x^2 + 3x + 3$

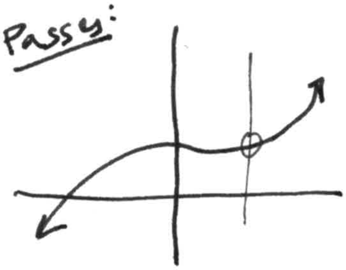
$\dfrac{f(1+h) - f(1)}{h} = \dfrac{[(1+h)^2 + (1+h) + 1] - [1^2 + 1 + 1]}{h}$

$= \dfrac{1 + 2h + h^2 + 1 + h + 1 - 3}{h}$

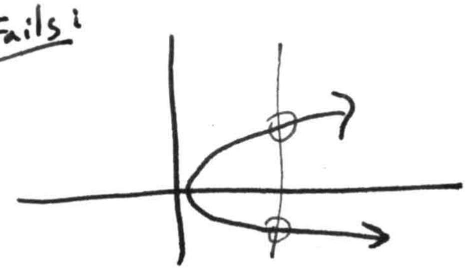
$= \dfrac{3h + h^2}{h} = 3 + h$

4. Sketch one graph that passes the vertical line test, and one that fails the vertical line test.

Passes:



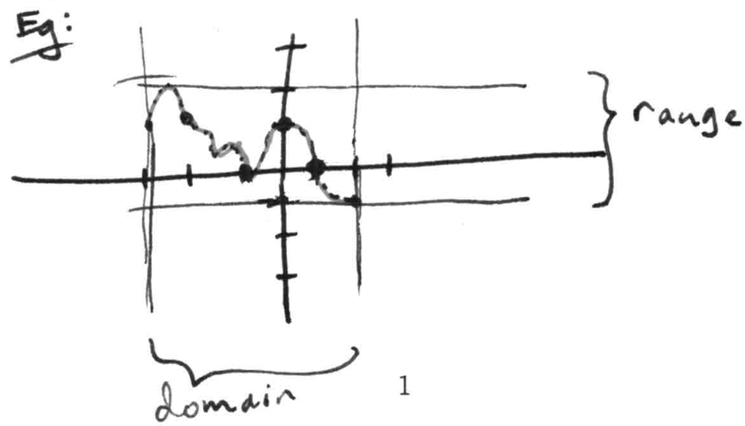
Fails:



5. Sketch the graph of a function with domain $[-3, 2]$ and range $[-1, 2]$ such that $f(-2) = 1$, $f(-1) = 0$, $f(0) = 1$, and $f(1) = 0$.

Your answers may vary a lot!

Eg:



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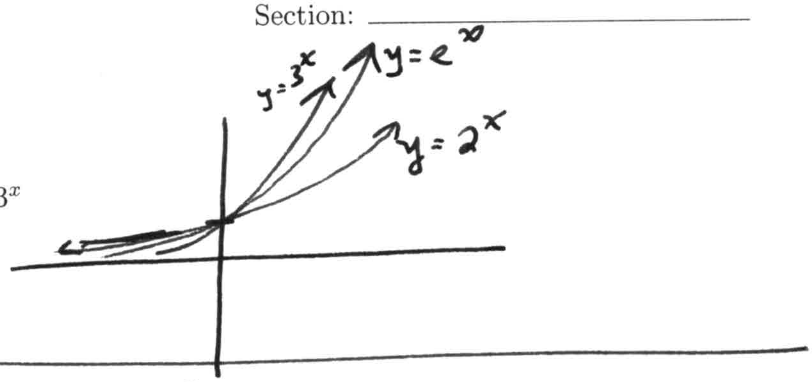
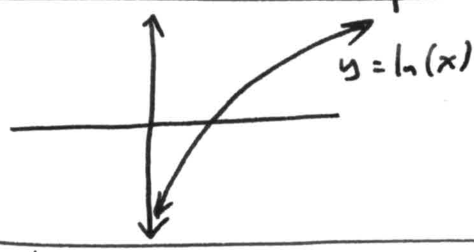
Exponentials and Logarithms

1. Sketch $y = e^x$, $y = 2^x$, and $y = 3^x$



2. Sketch $y = \ln(x)$



3. Simplify the expression:

$$\frac{(3y^2)^3}{y^4} = \frac{3^3 \cdot y^{2\cdot 3}}{y^4} = \frac{3^3 y^6}{y^4} = 3^3 y^2$$

4. Simplify the expression:

$$\frac{8^{-1/3}}{4^{-1/2}} = \frac{4^{\frac{1}{2}}}{8^{\frac{1}{3}}} = \frac{\sqrt{4}}{\sqrt[3]{8}} = \frac{2}{2} = 1$$

5. Simplify the expression:

$$\ln(x+1) + \ln(x-1) = \ln\big((x+1)(x-1)\big)$$
$$= \ln(x^2 - 1)$$

6. Simplify the expression:

$$3\ln(x) + 2\ln(x)$$
$$= \ln(x^3) + \ln(x^2)$$
$$= \ln(x^3 \cdot x^2) = \ln(x^5)$$

or

$$= 5 \cdot \ln(x)$$

Exam I Review - Worksheet   Math 1131   Class 16

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Limits and Continuity

1. Evaluate the following limit:

$$\lim_{x \to 3^-} \frac{x^2 - 3}{x + 3}$$

(this is continuous at 3 $\Rightarrow$ we can plug in 3)

$$= \text{[redacted]} \frac{(3)^2 - 3}{(3)+3} = \frac{9-3}{3+3} = \frac{6}{6} = 1$$

2. Evaluate the following limit:

$$\lim_{x \to 7} \frac{x^2 + \sqrt{5x} - e^x}{x^2 - 7x + 3}$$

(this is continuous at 7 $\Rightarrow$ we can plug in 7)

$$= \frac{7^2 + \sqrt{5 \cdot 7} - e^7}{7^2 - 7 \cdot 7 + 3} = \frac{7^2 + \sqrt{5 \cdot 7} - e^7}{3}$$

3. Evaluate the following limit:

$$\lim_{x \to -3} \frac{x^2 - 9}{x + 3}$$

(NOT continuous at -3 $\Rightarrow$ must do more work)

$$= \lim_{x \to -3} \frac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}} = \lim_{x \to -3} (x-3)$$

$$= -6$$

4. Evaluate the following limit:

$$\lim_{x \to -3} \frac{x^2 - 9}{x^2 + 5x + 6}$$

(NOT continuous at -3 $\Rightarrow$ must do more work)

$$= \lim_{x \to -3} \frac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}(x+2)} = \lim_{x \to -3} \frac{(x-3)}{(x+2)}$$

$$= \frac{-6}{-1} = 6$$

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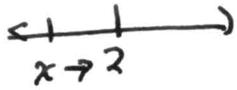
5. Evaluate the following limit:

$$\lim_{x\to 2^-} \frac{x+1}{x-2} = -\infty$$



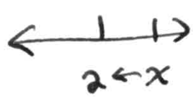
(NOT continuous at 2 and does NOT simplify ⇒ must think carefully)

as $x \to 2^-$, $(x-2)$ is ① negative & ② small so $\frac{1}{x-2}$ is ① negative ② big

so as $x \to 2^-$, $\frac{x+1}{x-2} \rightsquigarrow \frac{3}{x-2} = 3\left(\begin{matrix}\text{Big}\\ \text{Negative}\\ \#\end{matrix}\right)$ goes to $-\infty$

6. Evaluate the following limit:

$$\lim_{x\to 2^+} \frac{x+1}{x-2} = \infty$$



(NOT continuous at 2 & does NOT simplify)

Think as $x \to 2^+$, $x-2$ is ① positive ② small

so as $x \to 2^+$, $(x-2) \to 0^+$, so $\frac{1}{x-2} \to \infty$

AND as $x \to 2^+$, $(x+1) \to 3$, so $\frac{x+1}{x+2} \to \infty$

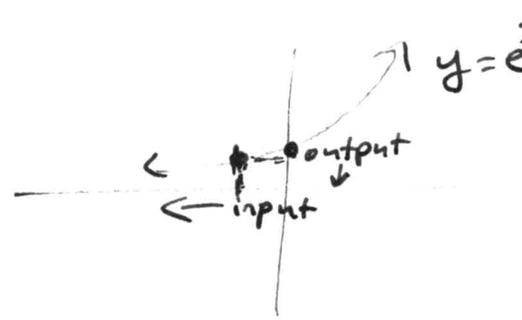
7. Evaluate the following limit:

$$\lim_{x\to 0^-} e^{1/x} = 0$$

as $x \to 0^-$ x is ① small ② negative

so $\frac{1}{x} \to -\infty$

as input $\to -\infty$, $e^{\text{input}} \to 0$



8. Evaluate the following limit:

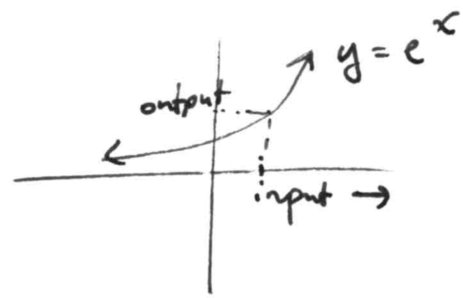
$$\lim_{x\to 0^+} e^{1/x} = \infty$$

as $x \to 0^+$

$\frac{1}{x} \to \infty$

as input $\to \infty$

$e^{\text{input}} \to \infty$



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9. Evaluate the following limit:

$$\lim_{x\to\infty}\frac{2x^3+3}{4x^2+7} = \lim_{x\to\infty}\frac{x^3\left(2+\frac{3}{x^3}\right)}{x^2\left(4+\frac{7}{x^2}\right)}$$

$$= \lim_{x\to\infty}\left[\underbrace{x\cdot\left(\frac{2+\frac{3}{x^3}}{4+\frac{7}{x^2}}\right)}_{\infty}\right] = \lim_{x\to\infty}\frac{2x}{4}$$

$$=\infty$$

10. Evaluate the following limit:

$$\lim_{x\to\infty}\frac{7x^2+4x}{3x^3+2x+7} = \lim_{x\to\infty}\frac{x^2\left(7+\frac{4}{x}\right)}{x^3\left(3+\frac{2}{x^2}+\frac{7}{x^3}\right)}$$

$$= \lim_{x\to\infty}\frac{\left(7+\frac{4}{x}\right)}{x\left(3+\frac{2}{x^2}+\frac{7}{x^3}\right)} = \lim_{x\to\infty}\frac{7}{3x} = 0$$

11. Evaluate the following limit:

$$\lim_{x\to\infty}\frac{2x^3+3}{3x^3+7} = \lim_{x\to\infty}\frac{x^3\left(2+\frac{3}{x^3}\right)}{x^3\left(3+\frac{7}{x^3}\right)}$$

$$= \lim_{x\to\infty}\frac{\left(2+\frac{3}{x^3}\right)}{\left(3+\frac{7}{x^3}\right)} = \frac{2}{3}$$

12. Evaluate the following limit:

$$\lim_{x\to\infty}\frac{2x^3+7x^2+4x+3}{3x^3+4x^2+2x+7}$$

13. Evaluate the following limit:

$$\lim_{x\to\infty}\frac{\sqrt{2x^3+1}}{4x^2+7}$$

14. Evaluate the following limit:

$$\lim_{x\to\infty}\frac{7x^2+4x}{\sqrt{3x^3+2x+7}}$$

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15. Find the vertical and horizontal asymptotes of the following function:

$$f(x) = e^{1/x}$$

only undefined at $x=0$

we've shown before $\lim_{x\to 0^+} e^{\frac{1}{x}} = \infty$

$\lim_{x\to 0^-} e^{\frac{1}{x}} = 0$

$\Rightarrow$ there is a vertical asymptote at $x=0$

to look for horizontal asymptotes,

$$\lim_{x\to\infty} e^{\frac{1}{x}} \left(\begin{matrix}\text{as } x\to\infty \\ \frac{1}{x}\to 0\end{matrix}\right) = \lim_{\boxed{\frac{1}{x}}\to 0} e^{\boxed{\frac{1}{x}}} = e^0 = 1$$

$$\lim_{x\to -\infty} e^{\frac{1}{x}} \left(\begin{matrix}\text{as } x\to -\infty \\ \frac{1}{x}\to 0\end{matrix}\right) = \lim_{\boxed{\frac{1}{x}}\to 0} e^{\boxed{\frac{1}{x}}} = e^0 = 1$$

$\Rightarrow$ there is a horizontal asymptote at $y=1$

16. Find the vertical and horizontal asymptotes of the following function:

$$f(x) = \frac{x+1}{x-2}$$

only undefined at $x=2$

& we've shown before

$\lim_{x\to 2^+} f(x) = \infty$

$\lim_{x\to 2^-} f(x) = -\infty$

$\Rightarrow$ there is a vertical asymptote at $x=2$

to find horizontal asymptote

$$\lim_{x\to\infty} \frac{x+1}{x-2} = \lim_{x\to\infty} \frac{\cancel{x}(1+\frac{1}{x})}{\cancel{x}(1-\frac{2}{x})} = 1$$

$$\text{and } \lim_{x\to -\infty} \frac{x+1}{x-2} = \lim_{x\to -\infty} \frac{\cancel{x}(1+\frac{1}{x})}{\cancel{x}(1-\frac{2}{x})} = 1$$

$\Rightarrow$ there is a horizontal asymptote at $y=1$

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Derivatives

Suppose $f(x)$ is some function.

1. Write down the limit definition of the derivative $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2. Explain the graphical meaning of the derivative $f'(x)$.

$f'(a)$ is the slope of the tangent to $f(x)$ at the point $(a, f(a))$

3. Write down an equation for the tangent line to $f(x)$ at the point $(a, f(a))$.

$$y = m(x - x_1) + y_1$$

$$y = f'(a)(x - a) + f(a)$$

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$\begin{array}{r} 32 \\ 4 \\ \hline 128 \end{array}$

4. Compute the derivative of the following function:

$$f(x) = 25x^5 + 32x^4 + 9x^3 + 144$$

$$f'(x) = \frac{d}{dx}\left(25x^5 + 32x^4 + 9x^3 + 144\right)$$
$$= \frac{d}{dx}(25x^5) + \frac{d}{dx}(32x^4) + \frac{d}{dx}(9x^3) + 144$$
$$= 125x^4 + 128x^3 + 27x^2 + 0$$

5. Compute the derivative of the following function:

$$f(x) = \sqrt{x} - 7e^x + (-1) \cdot x^9$$

6. Compute the derivative of the following function:

$(fg)' = fg' + gf'$

$$f(x) = (x^2 + x + 1) \cdot e^x$$

$$f'(x) = \frac{d}{dx}\left((x^2+x+1)e^x\right) = (x^2+x+1) \cdot \frac{d}{dx}\left[e^x\right] + e^x \cdot \frac{d}{dx}\left[x^2+x+1\right]$$
$$= (x^2+x+1)e^x + e^x(2x+1)$$
$$= e^x(x^2+3x+2)$$

7. Compute the derivative of the following function:

$$f(x) = e^x \cdot (\sqrt{x} + 1)$$

8. For each of the above, write the equation for the tangent line to $f(x)$ at the point $(1, f(1))$.

for # 4: $f(1) = 25 + 32 + 9 + 144 = 210$, $f'(1) = 280$ $\Rightarrow$ $y = 280(x-1) + 210$

for #6: $f(1) = (1+1+1)e^1 = 3e$, $f'(1) = e^1(1^2 + 3 \cdot 1 + 2) = 6e$ $\Rightarrow$ $y = 6e(x-1) + 3e$

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9. Compute the derivative of the following function:

$$\boxed{\left(\frac{t}{b}\right)' = \frac{b\cdot t' - t\cdot b'}{b^2}}$$

$$f(x) = \frac{e^x}{x}$$

$$f'(x) = \frac{d}{dx}\left(\frac{e^x}{x}\right) = \frac{x\cdot\frac{d}{dx}(e^x) - e^x\cdot\frac{d}{dx}(x)}{(x)^2}$$

$$= \frac{x\cdot e^x - e^x}{x^2} \quad \text{or} \quad = e^x\left(\frac{x-1}{x^2}\right)$$

10. Compute the derivative of the following function:

$$\boxed{\left(\frac{t}{b}\right)' = \frac{b\cdot t' - t\cdot b'}{b^2}}$$

$$f(x) = \frac{x^2+1}{x^2+x+1}$$

$$f'(x) = \frac{d}{dx}\left(\frac{x^2+1}{x^2+x+1}\right) = \frac{(x^2+x+1)\cdot\frac{d}{dx}(x^2+1) - (x^2+1)\cdot\frac{d}{dx}(x^2+x+1)}{(x^2+x+1)^2}$$

$$= \frac{\overbrace{(x^2+x+1)(2x)}^{2x^3+2x^2+2x} - \overbrace{(x^2+1)(2x+1)}^{(2x^3+x^2+2x+1)}}{(x^2+x+1)^2} = \frac{2x^3+2x^2+2x-(2x^3+x^2+2x\ldots}{(x^2+x+1)}$$

$$\boxed{f'(x) = \frac{x^2-1}{(x^2+x+1)}}$$

11. Compute the derivative of the following function:

$$\boxed{\left(\frac{t}{b}\right)' = \frac{b\cdot t' - t\cdot b'}{b^2}}$$

$$f(x) = \frac{x^2+1}{x^2-1}$$

$$f'(x) = \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right) = \frac{(x^2-1)\cdot\frac{d}{dx}(x^2+1) - (x^2+1)\cdot\frac{d}{dx}(x^2-1)}{(x^2-1)^2} =$$

$$= \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} = \frac{(2x^3-2x)-(2x^3+2x)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

12. For each of the above, write the equation for the tangent line to $f(x)$ at the point $(2, f(2))$.

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13. Compute the derivative of the following function:

constant coefficient

$$f(x) = 2\sin(x)\cdot\cos(x)$$

$$f'(x) = \frac{d}{dx}\left(\boxed{2}\sin(x)\cdot\cos(x)\right) = 2\cdot\frac{d}{dx}\left(\sin(x)\cdot\cos(x)\right)$$

$(f\cdot g)' = f\cdot g' + g f'$

$$= 2\cdot\left(\sin x\cdot\frac{d}{dx}(\cos(x)) + \cos(x)\cdot\frac{d}{dx}(\sin(x))\right)$$

$$= 2\left((\sin x)(-\sin(x)) + \cos(x)\cdot\cos(x)\right)$$

$$= 2\left(-\sin^2 x + \cos^2 x\right) = 2\cos^2 x - 2\sin^2 x$$

14. Compute the derivative of the following function:

$$\left(\frac{t}{b}\right)' = \frac{b\cdot t' - t b'}{b^2}$$

$$f(x) = \frac{\sin(x)}{x}$$

$$f'(x) = \frac{d}{dx}\left(\frac{\sin(x)}{x}\right) = \frac{x\cdot\frac{d}{dx}(\sin(x)) - \sin(x)\cdot\frac{d}{dx}(x)}{x^2}$$

$$= \frac{x\cdot\cos(x) - \sin(x)}{x^2} \quad \text{OR} \quad = \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}$$

Both are acceptable as final answers

15. Compute the derivative of the following function:

$$f(x) = \frac{\cos(x)+1}{\sin(x)-1}$$